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Basic theory
VEM for 2D Poisson equation

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1 Virtual Element Method for Poisson problem

1.1 Poisson equation

In this section a general Poisson equation is considered as

$$\begin{cases} \Delta u + f = 0 & \text{in} \quad \Omega \\ u = g_D & \text{on} \quad \Gamma_D \\ \partial_n u = g_N & \text{on} \quad \Gamma_N \end{cases} \tag{1}$$

where $\Omega \in \mathbb{R}$ is a polygonal domain and $f \in L^2(\Omega)$. The variational formulation reads

$$\begin{cases} \text{find} \quad u \in V := H_0^1(\Omega) \quad \text{such that} \\ a(u,v) = (f,v) := \mathcal{L}(v) \quad \forall v \in V \end{cases} \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}\Omega \tag{3}$$

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into K, and let $\mathcal{E}_h$ be the set of edges e of $\mathcal{T}_h$. Then the discrete problem becomes

$$\begin{cases} \text{find} \quad u_h \in V_h \quad \text{such that} \\ a_h(u_h,v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_h \subset V \end{cases} \tag{4}$$

1.2 Virtual element function spaces

Consider the first virtual element space

$$V_k(K) := \left\{ v \in H^1(K) : \Delta v \in \mathbb{P}_{k-2}(K) \quad \text{in} \quad K, \quad v|_{\partial K} = \mathbb{B}_k(\partial K) \right\} \tag{5}$$

where $\mathbb{P}_k$ is a polynomial with the highest order not exceeding k,

$$\mathbb{B}_k(\partial K) := \{ v \in C(\partial K) : v_e \in \mathbb{P}_k(e), \quad e \subset \partial K \} \tag{6}$$

It is not difficult to find that $\mathbb{B}_k(\partial K)$ is a linear space of dimension $n + n(k-1) = nk$, n is the number of sides of the polygon. Besides, the dimension of $V_k(K)$ is

$$\mathrm{dim} V_k(K) = N_K = n + n(k-1) + \frac{k(k-1)}{2} = n + \frac{k(k-1)}{2} \tag{7}$$

where the last term corresponds to the dimension of polynomials of degree $\leq k-2$ in two dimensions.

In $V_k(K)$, the degrees of freedom are selected as

- $\mathcal{V}_k(K)$: the values of v_h at the vertices;
- $\mathcal{E}_k(K)$: for $k > 1$, the values of v_h at $k-1$ uniformly spaced points on each edge e;
- $\mathcal{P}_k(K)$: for $k > 1$, the moments

$$\frac{1}{|K|}\int_K v_h m_\alpha \mathrm{d}\Omega, \quad \forall m \in \mathcal{M}_{k-2}(K)$$

In the last item, the $\mathcal{M}_{k-2}$ is the set of $(k^2-k)/2$ monomials

$$\mathcal{M}_{k-2} = \left\{ \left(\frac{\boldsymbol{x} - \boldsymbol{x}_K}{h_K} \right)^s, |\boldsymbol{s}| \leq k-2 \right\} \tag{8}$$

where h_K is the diameter of K, $\boldsymbol{x}_K$ is the centroid of K, $|K|$ is the area of the polygonal element. The above variables can be calculated by

$$|K| = \frac{1}{2}\left| \sum_{i=1}^{n} x_i y_{i+1} - x_{i+1} y_i \right| \tag{9}$$

Besides, the centroid (x_K, y_K) can be calculated by

$$x_K = \frac{1}{6|K|}\sum_{i=1}^{n}(x_i + x_{i+1})\left(x_i y_{i+1} - x_{i+1} y_i\right) \tag{10}$$

$$y_K = \frac{1}{6|K|}\sum_{i=1}^{n}(y_i + y_{i+1})\left(x_i y_{i+1} - x_{i+1} y_i\right) \tag{11}$$

Conventionally, $\mathcal{M}_r = 0$ for $r \leq -1$. For the multi-index $\boldsymbol{s} \in \mathbb{N}^d$, we follow the usual notation

$$\boldsymbol{x}^{\boldsymbol{s}} = x_1^{s_1} \cdots x_d^{s_d}, \quad |\boldsymbol{s}| = s_1 + \cdots + s_d \tag{12}$$

Note that $\mathcal{M}_{k-2}$ is a basis for $\mathbb{P}_{k-2}(K)$.

1.3 Projection operator and stability item

The projection operator

$$\Pi_k^K : V_k(K) \to \mathbb{P}_k(K) \tag{13}$$

represents the projection of any function in the local virtual element space $V_k(K)$ onto the subspace of linear polynomials. This projection is defined for $v \in V_k(K)$ by the conditions

$$\begin{cases} a^K(\Pi_k^K v, p) = a^K(v, p), \quad \forall q \in \mathbb{P}_k(K) \\ \overline{\Pi_k^K v} = \bar{v} \end{cases} \tag{14}$$

where

$$\overline{w_h} := \frac{1}{n}\sum_{i=1}^{n} w_h(v_i) \tag{15}$$

denotes the average value of w_h at the vertices for $k=1$. At the point, choosing $a_h^K(u,v) = a^K(\Pi_k^K u, \Pi_k^K v)$ would ensure the consistency property

$$a_h^K(p, v_h) = a^K(p, v_h) \tag{16}$$

Besides the consistency property, the stability property should also be satified, which described as

$$\forall v_h \in V_h(K), \quad \alpha_* a^K(v_h, v_h) \le a_h^K(v_h, v_h) \le \alpha^* a^K(v_h, v_h) \tag{17}$$

where α_* and α^* are two different constants. In order to satified the stability property, let $S^K(u,v)$ be any symmetric positive definite bilinear form to be chosen to verify

$$c_0 a^K(v,v) \le S^K(v,v) \le c_1 a^K(v,v) \quad \forall v \in V_k(K) \quad \text{with} \quad \Pi_k^K v = 0 \tag{18}$$

for some positive constants c_0 and c_1. Then set

$$a_h^K(u,v) = a^K\left(\Pi_k^K u, \Pi_k^K v\right) + S^K\left(u - \Pi_k^K u, v - \Pi_k^K v\right) \quad \forall u, v \in V^k(K) \tag{19}$$

easy to find that the bilinear form Eq.(19) satifies the consistency property Eq.(16) and the stability property Eq.(17).

1.4 Local Stiffness Matrix

$$a_h^K(u,v) = a^K\left(\Pi_k^K u, \Pi_k^K v\right) + S^K\left(u - \Pi_k^K u, v - \Pi_k^K v\right) \tag{20}$$

The basis functions $\phi_i \in V_k(K)$ are defined as usual as the canonical basis functions

$$\chi_i(\phi_j) = \mathrm{dof}_i(\phi_j) = \delta_{ij}, \quad i, j = 1, \cdots, N_K \tag{21}$$

so that a Lagrange-type interpolation identity can be obtained as

$$u_h = \sum_{i=1}^{N_K} \mathrm{dof}_i(v_h)\phi_i \quad \text{for all} \quad v_h \in V_k(K) \tag{22}$$

where $N_K := \mathrm{dim} V_k(K)$.

Based on the item discussed in section 1.3, the stiffness matrix is given by

$$a_h^K(u,v) = a^K\left(\Pi_k^K u, \Pi_k^K v\right) + S^K\left(u - \Pi_k^K u, v - \Pi_k^K v\right) \tag{23}$$

or

$$a_h^K(u,v) = \int_K \nabla\left(\Pi_k^K u\right) \cdot \nabla\left(\Pi_k^K v\right) \mathrm{d}\Omega + S^K\left(u - \Pi_k^K u, v - \Pi_k^K v\right) \tag{24}$$

Considering the interpolation in Eq.(22), the form of the local stiffness matrix can be obtained as

$$K_{i,j}(K) = a^K\left(\Pi_k^K \phi_i, \Pi_k^K \phi_j\right) + S^K\left(\phi_i - \Pi_k^K \phi_i, \phi_j - \Pi_k^K \phi_j\right) \tag{25}$$

1.4.1 Ritz projection

Since $\mathcal{M}_k(K)$ is a basis for $\mathbb{P}_k(K)$,the projection $\Pi_k^K \phi_i$ in Eq.(25) can be expanded in the basis of $\mathbb{P}_k(K)$ or in that of V_h^K:

$$\Pi_k^K \phi_i = \sum_{\alpha=1}^{N_{\mathbb{P}}} a_{\alpha,i} m_\alpha = \sum_{j=1}^{N_K} s_{j,i} \phi_j \tag{26}$$

The equation can be written in the matrix form as

$$\left[\Pi_k^K \phi_1, \Pi_k^K \phi_2, \cdots, \Pi_k^K \phi_{N_K}\right] = \Pi_k^K \boldsymbol{\phi}^T = \boldsymbol{m}^T \boldsymbol{\Pi}_{k*}^K = \boldsymbol{\phi}^T \boldsymbol{\Pi}_k^K \tag{27}$$

where $\boldsymbol{\Pi}_{k*}^K$ is the Ritz projection, and

$$\left(\boldsymbol{\Pi}_{k*}^K\right)_{i\alpha} = a_{i,\alpha}, \quad \left(\boldsymbol{\Pi}_k^K\right)_{ij} = s_{i,j} \tag{28}$$

Besides, Eq.(14) can be written as

$$\begin{cases} a^K\left(\Pi_k^K \phi_i, m_\alpha\right) = a^K\left(\phi_i, m_\alpha\right) \\ \overline{\Pi_k^K \phi_i} = \overline{\phi_i} \end{cases}, \quad i = 1, \cdots, N_K, \alpha = 1, \cdots, N_{\mathbb{P}} \tag{29}$$

or in matrix form as

$$\begin{cases} a^K\left(\boldsymbol{\Pi}_k^K \boldsymbol{\phi}, \boldsymbol{m}^T\right) = a^K\left(\boldsymbol{\Pi}_{k*}^K \boldsymbol{m}, \boldsymbol{m}^T\right) = a^K\left(\boldsymbol{\phi}, \boldsymbol{m}^T\right) \\ \overline{\boldsymbol{\Pi}_k^K \boldsymbol{\phi}^T} = \overline{\boldsymbol{\phi}^T} \end{cases} \tag{30}$$

Let

$$\boldsymbol{G} = a^K\left(\boldsymbol{m}, \boldsymbol{m}^T\right) = \int_K \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{m}^T \mathrm{d}\Omega \tag{31}$$

$$\boldsymbol{B} = a^K\left(\boldsymbol{m}, \boldsymbol{\phi}^T\right) = \int_K \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{\phi}^T \mathrm{d}\Omega \tag{32}$$

with the matrix form

$$\boldsymbol{G} = \begin{bmatrix} (\nabla m_1, \nabla m_1) & (\nabla m_1, \nabla m_2) & \cdots & (\nabla m_1, \nabla m_{N_{\mathbb{P}}}) \\ (\nabla m_2, \nabla m_1) & (\nabla m_2, \nabla m_2) & \cdots & (\nabla m_2, \nabla m_{N_{\mathbb{P}}}) \\ \vdots & \vdots & \ddots & \vdots \\ (\nabla m_{N_{\mathbb{P}}}, \nabla m_1) & (\nabla m_{N_{\mathbb{P}}}, \nabla m_2) & \cdots & (\nabla m_{N_{\mathbb{P}}}, \nabla m_{N_{\mathbb{P}}}) \end{bmatrix} \tag{33}$$

$$\boldsymbol{B} = \begin{bmatrix} (\nabla m_1, \nabla \phi_1) & (\nabla m_1, \nabla \phi_2) & \cdots & (\nabla m_1, \nabla \phi_{N_K}) \\ (\nabla m_2, \nabla \phi_1) & (\nabla m_2, \nabla \phi_2) & \cdots & (\nabla m_2, \nabla \phi_{N_K}) \\ \vdots & \vdots & \ddots & \vdots \\ (\nabla m_{N_{\mathbb{P}}}, \nabla \phi_1) & (\nabla m_{N_{\mathbb{P}}}, \nabla \phi_2) & \cdots & (\nabla m_{N_{\mathbb{P}}}, \nabla \phi_{N_K}) \end{bmatrix} \tag{34}$$

In the next, Eq.(30) can be written as

$$\begin{cases} \boldsymbol{G}\boldsymbol{\Pi}_{k*}^K = \boldsymbol{B} \\ \overline{\boldsymbol{m}}\boldsymbol{\Pi}_{k*}^K = \overline{\boldsymbol{\phi}} \end{cases} \tag{35}$$

It must be noted that the matrix $\boldsymbol{G}$ is not invertible because its first row is $\mathbf{0}$. Therefore, the first row of the matrix $\boldsymbol{G}$ can be replaced by the constraints of the projection:

$$\tilde{\boldsymbol{G}}\boldsymbol{\Pi}_{k*}^{K} = \tilde{\boldsymbol{B}} \tag{36}$$

where

$$\tilde{\boldsymbol{G}} = \boldsymbol{G} + \begin{bmatrix} \overline{\boldsymbol{m}} \\ 0 \end{bmatrix}, \tilde{\boldsymbol{B}} = \boldsymbol{B} + \begin{bmatrix} \overline{\boldsymbol{\phi}} \\ 0 \end{bmatrix} \tag{37}$$

Then, the Ritz projection can be calculated by

$$\boldsymbol{\Pi}_{k*}^{K} = \tilde{\boldsymbol{G}}^{-1}\tilde{\boldsymbol{B}} \tag{38}$$

1.4.2 Matrix of the projection operator

In order to obtain the matrix representation of the projection operator Π_k^K, let

$$\Pi_k^K \phi_i = \sum_{j=1}^{N_K} \mathrm{dof}_j\left(\Pi_k^K \phi_i\right)\phi_j = \sum_{j=1}^{N_K} s_{i,j}\phi_j, \quad i = 1, \cdots, N_K \tag{39}$$

Based on Eq.(26), we have

$$\Pi_k^K \phi_i = \sum_{\alpha=1}^{N_{\mathbb{P}}} a_{i,\alpha} m_\alpha = \sum_{\alpha=1}^{N_{\mathbb{P}}} a_{i,\alpha}\left(\sum_{j=1}^{N_K} \mathrm{dof}_j\left(m_\alpha\right)\phi_j\right) = \sum_{j=1}^{N_K}\left(\sum_{\alpha=1}^{N_{\mathbb{P}}} a_{i,\alpha}\mathrm{dof}_j(m_\alpha)\right)\phi_j \tag{40}$$

so that

$$s_{i,j} = \sum_{\alpha=1}^{N_{\mathbb{P}}} a_{i,\alpha}\mathrm{dof}_j(m_\alpha) \tag{41}$$

Defining matrix $\boldsymbol{D}$ with size $N_K \times N_{\mathbb{P}}$ by

$$\boldsymbol{D}_{j\alpha} := \mathrm{dof}_j(m_\alpha), \quad j = 1, \cdots, N_K, \quad \alpha = 1, \cdots, N_{\mathbb{P}} \tag{42}$$

with the matrix form as

$$\boldsymbol{D} = \begin{bmatrix} \mathrm{dof}_1\left(m_1\right) & \mathrm{dof}_1\left(m_2\right) & \cdots & \mathrm{dof}_1\left(m_{N_{\mathbb{P}}}\right) \\ \mathrm{dof}_2\left(m_1\right) & \mathrm{dof}_2\left(m_2\right) & \cdots & \mathrm{dof}_2\left(m_{N_{\mathbb{P}}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{dof}_{N_K}\left(m_1\right) & \mathrm{dof}_{N_K}\left(m_2\right) & \cdots & \mathrm{dof}_{N_K}\left(m_{N_{\mathbb{P}}}\right) \end{bmatrix} \tag{43}$$

and combining Eqs.(28), (38) and (41), we have

$$s_{i,j} = \sum_{\alpha=1}^{N_{\mathbb{P}}} \left(\boldsymbol{\Pi}_{k*}^{K}\right)_{i\alpha}\boldsymbol{D}_{j\alpha} = \sum_{\alpha=1}^{N_{\mathbb{P}}}\left(\tilde{\boldsymbol{G}}^{-1}\tilde{\boldsymbol{B}}\right)_{i\alpha}\boldsymbol{D}_{j\alpha} = \left(\boldsymbol{D}\tilde{\boldsymbol{G}}^{-1}\tilde{\boldsymbol{B}}\right)_{ji} \tag{44}$$

Hence, the matrix representation $\boldsymbol{\Pi}_k^K$ of the operator $\Pi_k^K : V_k(K) \to V_k(K)$ in the canonical basis is given by

$$\boldsymbol{\Pi}_k^K = \boldsymbol{D}\boldsymbol{\Pi}_{k*}^{K} = \boldsymbol{D}\tilde{\boldsymbol{G}}^{-1}\tilde{\boldsymbol{B}} \tag{45}$$

It is not necessary to calculated $\tilde{\boldsymbol{G}}$ in Eq.(45) because

$$\boldsymbol{G} = \int_K \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{m} \mathrm{d}\Omega = \int_K \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{\phi} \mathrm{d}\Omega \boldsymbol{D} = \boldsymbol{B}\boldsymbol{D} \tag{46}$$

so that

$$\begin{aligned} \tilde{\boldsymbol{G}} = \boldsymbol{G} + \begin{bmatrix} \overline{\boldsymbol{m}} \\ 0 \end{bmatrix} &= \int_K \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{m} \mathrm{d}\Omega + \begin{bmatrix} \overline{\boldsymbol{m}} \\ 0 \end{bmatrix} \\ &= \int_K \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{\phi} \mathrm{d}\Omega \boldsymbol{D} + \begin{bmatrix} \overline{\boldsymbol{m}} \\ 0 \end{bmatrix} = \tilde{\boldsymbol{B}}\boldsymbol{D} \end{aligned} \tag{47}$$

1.4.3 Stabilization term

Up to now, the unknown in Eq.(25) is the stabilization term. In general, the choice of the bilinear form S^K would depend on the problem and on the degrees of freedom. The stabilization term the following approximation

$$\begin{aligned} S^K\left(\phi_i - \Pi_k^K \phi_i, \phi_j - \Pi_k^K \phi_j\right) &= S^K\left(\phi_i - \Pi_k^K \phi_i, \phi_j - \Pi_k^K \phi_j\right) \\ &= a^E\left(\sum_{r=1}^{N_K} \mathrm{dof}_r\left(\phi_i - \Pi_k^K \phi_i\right)\phi_r, \sum_{r=1}^{N_K} \mathrm{dof}_r\left(\phi_j - \Pi_k^K \phi_j\right)\phi_r\right) \\ &= \sum_{r=1}^{N_K} \mathrm{dof}_r\left(\phi_i - \Pi_k^K \phi_i\right)\phi_r \mathrm{dof}_r\left(\phi_j - \Pi_k^K \phi_j\right)\phi_r a^K\left(\phi_r, \phi_r\right) \end{aligned} \tag{48}$$

Eqsy to find that we have $a^K\left(\phi_r, \phi_r\right) \simeq 1$ for all r, it will be sufficient to take the simple choice

$$S^K\left(\phi_i - \Pi_k^K \phi_i, \phi_j - \Pi_k^K \phi_j\right) = \sum_{r=1}^{N_K} \mathrm{dof}_r\left(\phi_i - \Pi_k^K \phi_i\right)\phi_r \mathrm{dof}_r\left(\phi_j - \Pi_k^K \phi_j\right)\phi_r \tag{49}$$

1.4.4 Last form of element stiffness metrix

The element stiffness matrix is written in Eq.(25):

$$\boldsymbol{K}_{ij} = a^K\left(\Pi_k^K \phi_i, \Pi_k^K \phi_j\right) + S^K\left(\phi_i - \Pi_k^K \phi_i, \phi_j - \Pi_k^K \phi_j\right) \tag{50}$$

Conventionally, we defined the following two items

$$\boldsymbol{K}_{ij}^1 = a^K\left(\Pi_k^K \phi_i, \Pi_k^K \phi_j\right) = \int_K \nabla\left(\Pi_k^K \phi_j\right) \cdot \nabla\left(\Pi_k^K \phi_i\right) \mathrm{d}\Omega \tag{51}$$

and

$$\begin{aligned} \boldsymbol{K}_{ij}^2 &= S^K\left(\phi_i - \Pi_k^K \phi_i, \phi_j - \Pi_k^K \phi_j\right) \\ &= \sum_{r=1}^{N_K} \mathrm{dof}_r\left(\phi_i - \Pi_k^K \phi_i\right)\phi_r \mathrm{dof}_r\left(\phi_j - \Pi_k^K \phi_j\right)\phi_r \end{aligned} \tag{52}$$

The matrix form of Eq.(51) is

$$
\begin{aligned}
\boldsymbol{K}_{ij}^{1} &= \int_K \nabla \left(\Pi_k^K \phi_j\right) \cdot \nabla \left(\Pi_k^K \phi_i\right) \mathrm{d}\Omega \\
&= \sum_{\alpha=1}^{N_{\mathbb{P}}} \sum_{\beta=1}^{N_{\mathbb{P}}} a_{i,\alpha} a_{j,\beta} \int_K \nabla m_\alpha \nabla m_\beta \mathrm{d}\Omega \\
&= \left(\left(\boldsymbol{\Pi}_{k*}^K\right)^T \boldsymbol{G} \boldsymbol{\Pi}_{k*}^K \right)_{ij}
\end{aligned} \tag{53}
$$

For Eq.(39), we know that

$$
\mathrm{dof}_r \left(\Pi_k^K \phi_i\right) = \left(\boldsymbol{\Pi}_k^K\right)_{ri} \tag{54}
$$

so that the matrix form of Eq.(52) can be obtained as

$$
\boldsymbol{K}_{ij}^{2} = \sum_{r=1}^{N_K} \left(\boldsymbol{I} - \boldsymbol{\Pi}_k^K\right)_{ri} \left(\boldsymbol{I} - \boldsymbol{\Pi}_k^K\right)_{rj} = \left[\left(\boldsymbol{I} - \boldsymbol{\Pi}_k^K\right)^T \left(\boldsymbol{I} - \boldsymbol{\Pi}_k^K\right) \right]_{ij} \tag{55}
$$

We end up with the following matrix expression for the VEM local stiffness matrix:

$$
\boldsymbol{K}_k^K = \left(\boldsymbol{\Pi}_{k*}^K\right)^T \boldsymbol{G} \boldsymbol{\Pi}_{k*}^K + \left(\boldsymbol{I} - \boldsymbol{\Pi}_k^K\right)^T \left(\boldsymbol{I} - \boldsymbol{\Pi}_k^K\right) \tag{56}
$$

1.5 VEM matrix calculation

For $k = 1$, the basis for the space $\mathbb{P}$ is selected as

$$
\mathcal{M}_K := \left\{ m_1(x,y) := 1, m_2(x,y) := \frac{x - x_K}{h_K}, m_3(x,y) := \frac{y - y_K}{h_K} \right\} \tag{57}
$$

so that the matrix $\boldsymbol{D}$ defined in Eq.(43) can be written as

$$
\boldsymbol{D} = \begin{bmatrix}
1 & \frac{x(1)-x_K}{h_K} & \frac{y(1)-y_K}{h_K} \\
1 & \frac{x(2)-x_K}{h_K} & \frac{y(2)-y_K}{h_K} \\
\vdots & \vdots & \vdots \\
1 & \frac{x(n)-x_K}{h_K} & \frac{y(n)-y_K}{h_K}
\end{bmatrix} \tag{58}
$$

The matrix $\boldsymbol{B}$ defined in Eq.(32)

$$
\begin{aligned}
\boldsymbol{B} &= \int_K \nabla \boldsymbol{m} \cdot \nabla \boldsymbol{\phi} \mathrm{d}\Omega \\
&= -\int_K \Delta \boldsymbol{m} \cdot \boldsymbol{\phi} \mathrm{d}\Omega + \sum_{e \subset \partial K} \int_e \left(\nabla \boldsymbol{m} \cdot \boldsymbol{n}_e\right) \boldsymbol{\phi} \mathrm{d}\Gamma
\end{aligned} \tag{59}
$$

For $k = 1$, easy to find that $\Delta \boldsymbol{m} = \boldsymbol{0}$ and $\nabla \boldsymbol{m}$ is a constant vector. Then the metrix $\boldsymbol{B}$ can be rewritten as

$$
\boldsymbol{B} = \nabla \boldsymbol{m} \sum_{e \subset \partial K} \int_e \boldsymbol{n}_e \boldsymbol{\phi} \mathrm{d}\Gamma, \quad \text{for} \quad k = 1 \tag{60}
$$

Then, based on Eqs.(46) and (47), the matrix $\boldsymbol{G}$ and $\tilde{\boldsymbol{G}}$ can be calculated by

$$\boldsymbol{G} = \boldsymbol{B}\boldsymbol{D}, \quad \tilde{\boldsymbol{G}} = \tilde{\boldsymbol{B}}\boldsymbol{D} \tag{61}$$

where $\tilde{\boldsymbol{B}}$ is constructed in Eq.(37). Based on Eqs.(38) and (45), the projection matrices can be calculated by

$$\boldsymbol{\Pi}_{k*}^{K} = \tilde{\boldsymbol{G}}^{-1}\tilde{\boldsymbol{B}}, \quad \boldsymbol{\Pi}_{k}^{K} = \boldsymbol{D}\boldsymbol{\Pi}_{k*}^{K} \tag{62}$$

Lastly, the element Stiffness matrix can be obtained as

$$\boldsymbol{K}_{k}^{K} = \left(\boldsymbol{\Pi}_{k*}^{K}\right)^{T} \boldsymbol{G}\boldsymbol{\Pi}_{k*}^{K} + \left(\boldsymbol{I} - \boldsymbol{\Pi}_{k}^{K}\right)^{T} \left(\boldsymbol{I} - \boldsymbol{\Pi}_{k}^{K}\right) \tag{63}$$

2 Some examples

For the Poisson equation given as

$$\Delta u = 0, \quad \text{in} \quad \Omega \tag{64}$$

with the boundary conditions described as

$$\begin{cases} u = 0 & \text{on} \quad x = 0 \\ u = 1 & \text{on} \quad x = L \end{cases} \tag{65}$$

two different geometric models are analyzed: a square and a logo of IKM. For the first model the size is selected as $x \times y = 1 \times 1 (L = 1)$. For the second model the size is selected as $x \times y = 100 \times 60 (L = 100)$.

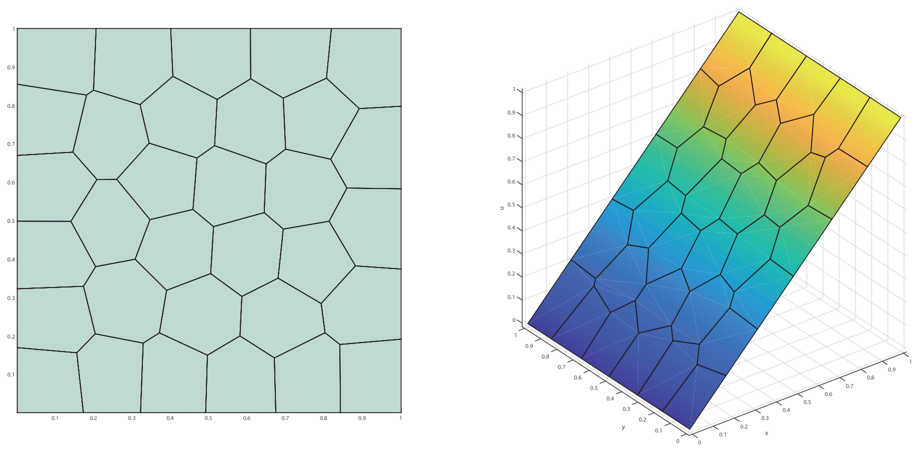

Figure 1: Contour plot of a square calculated by VEM.