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Basic theory

VEM for 2D elastic mechanics problems

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1 Problem statement

1.1 Governing equation

Considering the problems in solid mechanics defined in domain $\Omega \subseteq \mathbb{R}^d$ with boundary $\partial\Omega = \Gamma(d$ is the dimension), the strong form of the boundary value problem of elasticity is given by:

Find $\boldsymbol{u}(\boldsymbol{x}) : \bar{\Omega} \to \mathbb{R}^d$ such that

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = 0, \quad \boldsymbol{x} \in \Omega, \tag{1a}$$

$$\boldsymbol{u} = \boldsymbol{u}_D, \quad \boldsymbol{x} \in \Gamma_D, \tag{1b}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n}_N = \bar{\boldsymbol{t}}, \quad \boldsymbol{x} \in \Gamma_N. \tag{1c}$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ follows Hooke's low

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}, \tag{2}$$

where

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right). \tag{3}$$

1.2 Continuous variation problem

Assuming

$$\boldsymbol{\mathcal{V}} = \left\{\boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma_0\right\}, \tag{4}$$

the weak form of the governing equation is: find $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}$ such that

$$\int_\Omega -\frac{\partial\sigma_{ij}(\boldsymbol{u})}{\partial x_j} v_i \mathrm{d}\Omega = \int_\Omega f_i v_i \mathrm{d}\Omega. \tag{5}$$

By using integration by parts, we have

$$-\int_{\partial\Omega} \sigma_{ij}(\boldsymbol{u}) v_i n_j \mathrm{d}\Gamma + \int_\Omega \sigma_{ij}(\boldsymbol{u}) \frac{\partial v_i}{\partial x_j} \mathrm{d}\Omega = \int_\Omega f_i v_i \mathrm{d}\Omega. \tag{6}$$

Considering the symmetry of stress tension, we have

$$\sigma_{ij}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j} = \sigma_{ij}(\boldsymbol{u})\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v}), \tag{7}$$

and the last form can be written as

$$\int_{\Omega} \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\mathrm{d}\Omega = \int_{\Omega} f_i v_i \mathrm{d}\Omega + \int_{\Gamma_N} \bar{t}_i v_i \mathrm{d}\Gamma, \tag{8}$$

or the matrix form as

$$\int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\mathrm{d}\Omega = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\mathrm{d}\Omega + \int_{\Gamma_N} \bar{\boldsymbol{t}} \cdot \boldsymbol{v}\mathrm{d}\Gamma. \tag{9}$$

Then the variation problem can be written as: find $\boldsymbol{u} \in \boldsymbol{\mathcal{V}}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v}), \boldsymbol{v} \in \boldsymbol{\mathcal{V}}, \tag{10}$$

where

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\mathrm{d}\Omega, \tag{11}$$

$$L(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\mathrm{d}\Omega + \int_{\Gamma_N} \bar{\boldsymbol{t}} \cdot \boldsymbol{v}\mathrm{d}\Gamma. \tag{12}$$

1.3 Mathematical preliminaries

It is more convenient to reduce the tensor expressions into equivalent matrix and vector representations. In particular, for any symmetric 2×2 matrix $\boldsymbol{A}$, denote its Voigt representation $\bar{\boldsymbol{A}}$ by

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \bar{\boldsymbol{A}} = \begin{Bmatrix} a_{11} \\ a_{22} \\ a_{12} \end{Bmatrix}. \tag{13}$$

On using Voigt (engineering) notation, we can write the stress and strain in terms of 3×1 arrays:

$$\bar{\boldsymbol{\sigma}} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}, \quad \bar{\boldsymbol{\varepsilon}} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}. \tag{14}$$

Furthermore, by using these conventions we can also express the strain- displacement relation and the constitutive law in matrix form as:

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{C}\bar{\boldsymbol{\varepsilon}}, \quad \bar{\boldsymbol{\varepsilon}} = \boldsymbol{S}\boldsymbol{u}, \tag{15}$$

where $\boldsymbol{S}$ is a matrix differential operator that is given by

$$\boldsymbol{S} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \tag{16}$$

and $\boldsymbol{C}$ is the associated matrix representation of the material tensor that is given by

$$\boldsymbol{C} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}, \tag{17}$$

for plane stress and

$$\boldsymbol{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}, \tag{18}$$

for plane strain. Besides, E is Young's modulus and ν of the Poisson's ratio of the material.

2 Projection operator of the displacement-for strain

2.1 Projection operator

The projection is designed as

$$\Pi_{E,k} : \boldsymbol{\mathcal{V}}^h(E) \to \boldsymbol{\mathcal{P}}_k(E), \quad \boldsymbol{\mathcal{V}}^h(E) \equiv \left[\mathcal{V}^h(E)\right]^2. \tag{19}$$

The operator is constructed locally in each polygon so that satisfies the following orthogonality condition

$$a_E\left(\boldsymbol{v}^h - \Pi_k \boldsymbol{v}^h, \boldsymbol{p}\right) = 0. \tag{20}$$

The basis functions for space $\boldsymbol{\mathcal{P}}_k$ are selected as

- $k = 0$:

$$\boldsymbol{m}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \tag{21}$$

- $k = 1$:

$$\boldsymbol{m}_1 = \boldsymbol{m}_0, \begin{pmatrix} -\eta \\ \xi \end{pmatrix}, \begin{pmatrix} \eta \\ \xi \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tag{22}$$

where

$$\xi = \frac{x - x_E}{h_E}, \quad \eta = \frac{y - y_E}{h_E}, \tag{23}$$

and (x_E, y_E) is the center of the element and h_E is the characteristic length of the element.

The definition of the local projection operator $\Pi_k \equiv \Pi_{E,k}$ can be rewritten as

$$a_E(\boldsymbol{v}^h, \boldsymbol{p}) = a_E\left(\Pi_k \boldsymbol{v}^h, \boldsymbol{p}\right), \quad \forall \boldsymbol{p} \in \boldsymbol{\mathcal{P}}_k(E). \tag{24}$$

Furthermore, coefficients of the projection of shape functions are determined due to the orthogonal property

$$\begin{aligned} \int_E \boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{\phi}^T)\mathrm{d}\Omega &= \int_E \boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{\phi}^T\boldsymbol{\Pi})\mathrm{d}\Omega \\ &= \int_E \boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{m}^T\boldsymbol{\Pi}_k^*)\mathrm{d}\Omega \\ &= \int_E \boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{m}^T)\mathrm{d}\Omega\boldsymbol{\Pi}_k^* \end{aligned} \tag{25}$$

or

$$a_E\left(\boldsymbol{m},\boldsymbol{\phi}^T\right)=a_E\left(\boldsymbol{m},\boldsymbol{\phi}^T\boldsymbol{\Pi}_k\right)=a_E\left(\boldsymbol{m},\boldsymbol{m}^T\boldsymbol{\Pi}_k^*\right)=a_E\left(\boldsymbol{m},\boldsymbol{m}^T\right)\boldsymbol{\Pi}_k^*. \tag{26}$$

Lastly, the matrix of the projection operator can be written as

$$\begin{cases}\boldsymbol{G}\boldsymbol{\Pi}_k^*=\boldsymbol{B}\\ \text{constraints}\end{cases}, \tag{27}$$

where

$$\boldsymbol{G}=a_E\left(\boldsymbol{m},\boldsymbol{m}^T\right)=\int_E\boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{m}^T)\mathrm{d}\Omega, \tag{28}$$

$$\boldsymbol{B}=a_E\left(\boldsymbol{m},\boldsymbol{\phi}^T\right)=\int_E\boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{\phi}^T)\mathrm{d}\Omega, \tag{29}$$

and

$$\boldsymbol{G}=\boldsymbol{B}\boldsymbol{D},\quad \boldsymbol{\Pi}_k=\boldsymbol{D}\boldsymbol{\Pi}_k^*, \tag{30}$$

where

$$\boldsymbol{D}_{j\alpha}=\mathrm{dof}_j(\boldsymbol{m}_\alpha). \tag{31}$$

The matrix $\boldsymbol{B}$ can be calculated by

$$\begin{aligned}\boldsymbol{B}&=\int_E\boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{\phi}^T)\mathrm{d}\Omega\\ &=-\int_E\nabla\cdot(\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{m}))\cdot\boldsymbol{\phi}^T\mathrm{d}\Omega+\int_{\partial E}\boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{n}\boldsymbol{\phi}^T\mathrm{d}\Gamma,\end{aligned} \tag{32}$$

where the first term is zero and

$$\boldsymbol{n}=\begin{bmatrix}n_x&0\\0&n_y\\n_y&n_x\end{bmatrix}. \tag{33}$$

As mentioned in Eq.(27), the constraints should be introduced as

$$\begin{cases}\displaystyle\int_E\nabla\times\Pi_k^1\boldsymbol{v}\mathrm{d}\Omega=\int_E\nabla\times\boldsymbol{v}\mathrm{d}\Omega\\ \displaystyle\int_{\partial E}\Pi_k^1\boldsymbol{v}\mathrm{d}\Gamma=\int_{\partial E}\boldsymbol{v}\mathrm{d}\Gamma\end{cases}, \tag{34}$$

where

$$\int_E\nabla\times\boldsymbol{v}\mathrm{d}\Omega=\int_{\partial E}\boldsymbol{v}\cdot\boldsymbol{t}_e\mathrm{d}\Gamma, \tag{35}$$

where

$$\boldsymbol{t}_e=[-n_y,n_x]^T. \tag{36}$$

For the first term in Eq.(34), the right hand can be written as

$$\int_E\nabla\times\boldsymbol{\phi}^T\mathrm{d}\Omega=\int_{\partial E}\boldsymbol{t}_e^T\boldsymbol{\phi}^T\mathrm{d}\Gamma=\int_{\partial E}\begin{bmatrix}-n_y&n_x\end{bmatrix}\begin{bmatrix}\phi^T&\\&\phi^T\end{bmatrix}\mathrm{d}\Gamma. \tag{37}$$

For the second term in Eq.(34), we have

$$\int_{\partial E} \boldsymbol{\phi}^T \mathrm{d}\Gamma = \int_{\partial E} \begin{bmatrix} \phi^T & \\ & \phi^T \end{bmatrix} \mathrm{d}\Gamma. \tag{38}$$

Lastly, we have

$$\tilde{\boldsymbol{G}} = \tilde{\boldsymbol{B}}\boldsymbol{D}, \tag{39}$$

and the projection can be obtained as

$$\boldsymbol{\Pi}_k^* = \tilde{\boldsymbol{G}}^{-1}\tilde{\boldsymbol{B}}, \quad \boldsymbol{\Pi}_k = \boldsymbol{D}\boldsymbol{\Pi}_k^* = \boldsymbol{D}\tilde{\boldsymbol{G}}^{-1}\tilde{\boldsymbol{B}}. \tag{40}$$

2.2 Element stiffness matrix

The stiffness matrix is obtained by the contributions of consistency and stability components

$$\begin{aligned} \boldsymbol{K}_E &= \int_E \boldsymbol{\varepsilon}\left(\Pi\boldsymbol{\phi}\right)\boldsymbol{C}\boldsymbol{\varepsilon}\left(\Pi\boldsymbol{\phi}^T\right)\mathrm{d}\Omega + \boldsymbol{K}_E^s \\ &= \int_\Omega \boldsymbol{\Pi}_k^{*T}\boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}\left(\boldsymbol{m}^T\right)\boldsymbol{\Pi}_k^*\mathrm{d}\Omega + \boldsymbol{K}_E^s \\ &= \boldsymbol{\Pi}_k^{*T}\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{m})\boldsymbol{C}\boldsymbol{\varepsilon}\left(\boldsymbol{m}^T\right)\mathrm{d}\Omega\boldsymbol{\Pi}_k^* + \boldsymbol{K}_E^s \\ &= \boldsymbol{\Pi}_k^{*T}\boldsymbol{G}\boldsymbol{\Pi}_k^* + \boldsymbol{K}_E^s, \end{aligned} \tag{41}$$

and the stability component $\boldsymbol{K}_E^s$ can be selected as

$$\boldsymbol{K}_E^s = \tau^h \mathrm{tr}\left(\boldsymbol{K}_E^c\right)\left(\boldsymbol{I} - \boldsymbol{\Pi}_k\right)^T\left(\boldsymbol{I} - \boldsymbol{\Pi}_k\right). \tag{42}$$

2.3 Numerical example

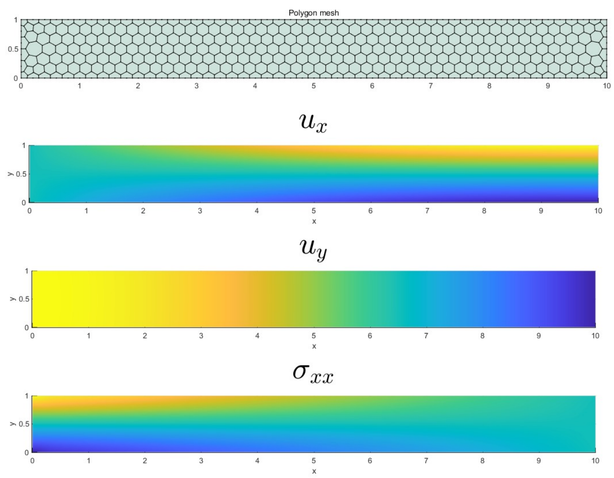

Figure 1: Numerical solutions obtained by VEM.